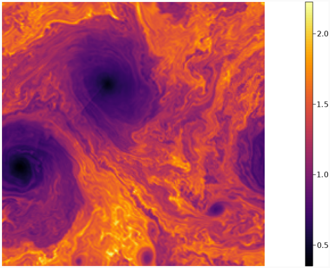
Entropy stable reduced order modeling of nonlinear conservation laws using high order DG methods

Ray Qu, Jesse Chan

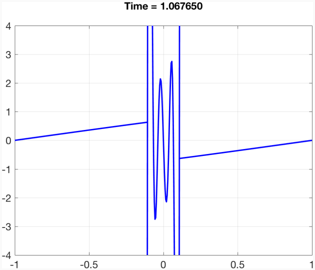
Dept. of Computational Applied Mathematics and Operations Research
Rice University

2024 Finite Element Rodeo

Why entropy stable reduced order modeling (ROM)?



(a) Kelvin–Helmholtz instability



(b) 1D Burgers'

- Extreme-scale nonlinear evaluations in high-fidelity simulations.
- ROM enables efficient many-query contexts.
- High order methods blow up around shocks and turbulence for both full order models (FOMs) and ROMs.

Entropy stability for nonlinear problems

- Energy balance for nonlinear conservation laws (Burgers', shallow water, compressible Euler + Navier-Stokes).

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_i^d \frac{\partial \mathbf{f}_i(\mathbf{u})}{\partial \boldsymbol{x}_i} = 0. \tag{1}$$

- Continuous entropy inequality: convex entropy function $S(\mathbf{u})$, "entropy potential" $\psi(\mathbf{u})$, entropy variables $\mathbf{v}(\mathbf{u})$

$$\int_\Omega \mathbf{v}^T \left(\frac{\partial \mathbf{u}}{\partial t} + \sum_i^d \frac{\partial \mathbf{f}_i(\mathbf{u})}{\partial \boldsymbol{x}_i} \right) = 0, \qquad \boxed{\mathbf{v}(\mathbf{u}) = \frac{\partial S}{\partial \mathbf{u}}}$$

$$\Longrightarrow \int_\Omega \frac{\partial S(\mathbf{u})}{\partial t} + \sum_i^d \left(\mathbf{v}^T \mathbf{f}_i(\mathbf{u}) - \psi_i(\mathbf{u}) \right) \Big|_{-1}^{1} \leq 0.$$

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Talk outline

1. Full order model (FOM) construction
2. Reduced order model (ROM) construction
3. Numerical Experiments

Full order model (FOM) construction

Entropy stable high order DG formulation

- A global DG formulation of (1) is

$$\mathbf{M}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \sum_{i=1}^{d}\left(2\left(\mathbf{Q}^i \circ \mathbf{F}^i\right)\mathbf{1} + \mathbf{E}^T\mathbf{B}^i\left(\mathbf{f}^{i,*} - \mathbf{f}^i(\mathbf{u})\right)\right) = \text{dissipation},$$

where $(\mathbf{F}^i)_{j,k} = \mathbf{f}^i(\mathbf{u}_j, \mathbf{u}_k)$ is a matrix of nonlinear flux evaluations, $\mathbf{E}$ is a boundary extraction matrix, and $\mathbf{Q}^i$ is a global summation-by-parts (SBP) operator $(\mathbf{Q}^i + (\mathbf{Q}^i)^T = \mathbf{E}^T\mathbf{B}^i\mathbf{E})$ with zero row sum $(\mathbf{Q}^i\mathbf{1} = \mathbf{0})$.

- We can prove a semi-discrete entropy stability condition

$$\mathbf{1}^T\mathbf{M}\frac{\mathrm{d}S(\mathbf{u})}{\mathrm{d}t} \leq 0.$$

Tadmor (1987). *The numerical viscosity of entropy stable schemes for systems of conservation laws.*

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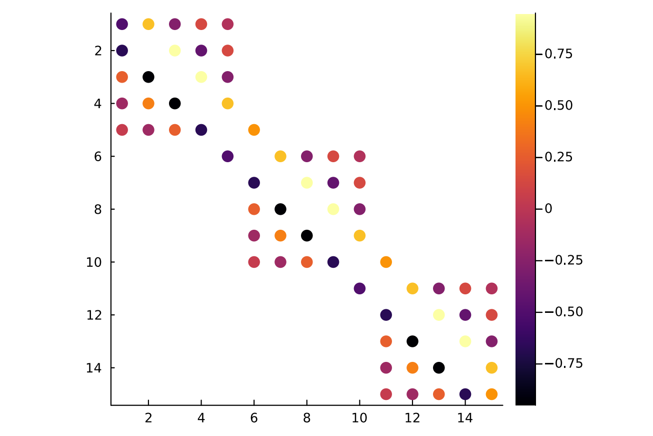
An example of a global SBP operator



Spy plot of $\mathbf{Q}^1$ (1D domain, 3 elements with 5 nodes in each)

Reduced order model (ROM) construction

Galerkin projection ROM

- Galerkin projection ROM ($\mathbf{V}_N$ is the POD basis):

$$\mathbf{M}_N \frac{\mathrm{d}\mathbf{u}_N}{\mathrm{dt}} + \sum_{i=1}^{d} \left(2\mathbf{V}_N^T (\mathbf{Q}^i \circ \mathbf{F}^i)\mathbf{1} + \mathbf{V}_b^T \mathbf{B}^i (\mathbf{f}^{i,\star} + \mathbf{f}^i(\mathbf{u}_N)) \right) = \mathbf{0},$$

  $\mathbf{u} \approx \mathbf{V}_N \mathbf{u}_N$ and $\mathbf{V}_b = \mathbf{E}\mathbf{V}_N$.

- To achieve entropy stability, use entropy projection

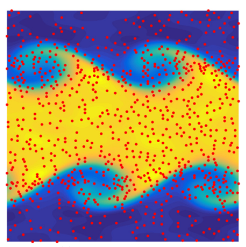
$$\widetilde{\mathbf{u}} = \mathbf{u}(\mathbf{V}_N \mathbf{V}_N^{\dagger} \mathbf{v}(\mathbf{V}_N \mathbf{u}_N)) = \mathbf{u}(\widetilde{\mathbf{v}}), \qquad (\mathbf{F}^i)_{j,k} = \mathbf{f}^i(\widetilde{\mathbf{u}}_j, \widetilde{\mathbf{u}}_k).$$

- Still has high computational cost! Needs hyper-reduction. Hyper-reduction on volume and boundary terms are independent.

Hyper-reduction on volume terms

Main idea: find a set of volume nodes and positive weights $(I_v, \mathbf{w}_v)$

$$\mathbf{V}_N^T g(\mathbf{V}_N \mathbf{u}_N) \approx \mathbf{V}_N(I_v, :)^T \mathrm{diag}(\mathbf{w}_v) g(\mathbf{V}_N(I_v, :)\mathbf{u}_N).$$



We use the greedy algorithm for empirical cubature.

Chan (2020). *Entropy stable reduced order modeling of nonlinear conservation laws.*

Hernádez et al. (2017) *Dimensional hyper-reduction of nonlinear finite element models via empirical cubature.*

Hyper-reduction on boundary terms

Assume hyper-reduction on boundary $(I_b, \mathbf{w}_b)$ and denote

$$\bar{\mathbf{B}}^i = \text{diag}(\mathbf{n}^i)\text{diag}(\mathbf{w}_b).$$

Suppose $\mathbf{V}^i$ is some ROM basis matrix for the ith coordinate,

$$\mathbf{1}^T \mathbf{Q}^i \mathbf{V}^i = \mathbf{1}^T \bar{\mathbf{B}}^i \mathbf{V}^i(I_b, :),$$

is a matrix form of the fundamental theorem of calculus, and we need to enforce this equality to preserve entropy stability.

A natural way to do this: Carathéodory's pruning.

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Carathéodory's pruning

Carathéodory's theorem states that, for any M-point positive quadrature rule exact on space $\mathbf{V}$ with $\dim(\mathbf{V}) = N$, we can always generate a new N-point interpolatory positive rule to preserve all moments.

In our case,

$$\mathbf{1}^T \mathbf{B}^i \mathbf{V}^i = \int \phi_j^i(\mathbf{x}_k)\mathbf{n}^i = \sum_{k=1}^{M} \mathbf{w}_{b,j} \mathbf{n}_j^i \phi_j(\mathbf{x}_k).$$

In practice, we concatenate all dimensions

$$\begin{bmatrix} \text{diag}(\mathbf{n}^1)\mathbf{V}^1 & \cdots & \text{diag}(\mathbf{n}^d)\mathbf{V}^d \end{bmatrix},$$

which yields $\mathcal{O}(dN)$ hyper-reduced positive boundary weights $\mathbf{w}_b$ and node indices I_b.

Courtesy of Dr. Akil Narayan (U. Utah)

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Numerical Experiments

Example 1 - 1D Euler

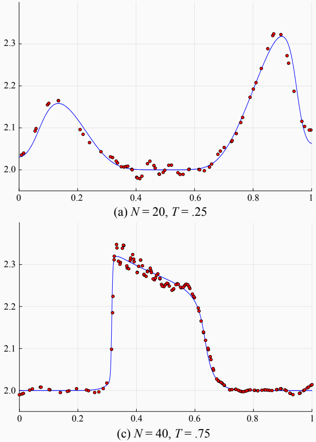

Figure 1: 1D Compressible Euler (reflective wall). FOM dim: 2048. Viscosity: 2×10^{-4}. Runtime $T = .75$.

Example 2 - sod shock tube

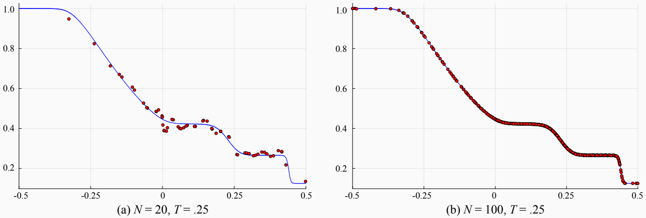

Figure 2: FOM dim: 2048. Viscosity: 2×10^{-3}. Runtime $T = .25$.

Example 3 - Gaussian

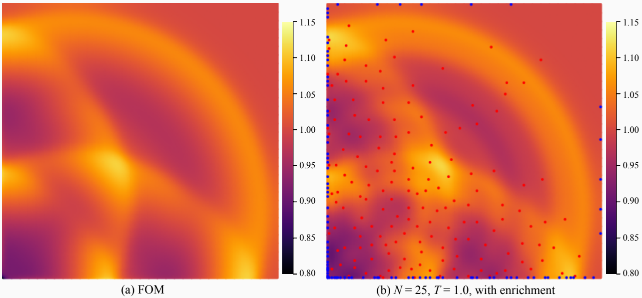

Figure 3: 2D compressible Euler (reflective wall). FOM dim: 6400. Viscosity: 1×10^{-3}. Run time $T = 1.0$. Boundary hyper-reduced by Carathéodory pruning.

Summary and acknowledgement

In this work, we

- present an entropy stable reduced order modeling of nonlinear conservation laws based on high order DG methods.
- develop structure-preserving hyper-reduction techniques (Carathéodory pruning) which preserve entropy stability.

This work was supported by NSF grant DMS-1943186.