Entropy stable reduced order modeling of nonlinear conservation laws using discontinuous Galerkin methods

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Abstract

- Extension of ES-ROMs for nonlinear conservation laws from finite volume methods [1] to high order DG methods.
- ES-ROMs are stable independently of accuracy performance.

Background

- Nonlinear conservation laws with conservative variables $\boldsymbol{u} \in \mathbb{R}^n$ on domain Ω

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{i=1}^{d} \frac{\partial \boldsymbol{f}^i(\boldsymbol{u})}{\partial \boldsymbol{x}^i} = 0, \qquad (\boldsymbol{x}, t) \in \Omega \times [0, \infty). \tag{1}$$

- (1) admits entropy stability on continuous level with convex entropy function $S(\boldsymbol{u})$:

$$\int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} d\boldsymbol{x} + \sum_{i=1}^{d} \int_{\partial\Omega} \left(\boldsymbol{v}^T \boldsymbol{f}^i(\boldsymbol{u}) - \psi^i(\boldsymbol{u})\right) \boldsymbol{n}^i \leqslant 0. \tag{2}$$

Reduced order modeling

- The global DG formulation of (1) is

$$\boldsymbol{M} \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \sum_{i=1}^{d} \left(2\left(\boldsymbol{Q}^i \circ \boldsymbol{F}^i\right)\boldsymbol{1} + \boldsymbol{B}^i \boldsymbol{f}^{i,*}\right) = \boldsymbol{0}, \tag{3}$$

where $\left(\boldsymbol{F}^i\right)_{j,k} = \boldsymbol{f}_i(\boldsymbol{u}_j, \boldsymbol{u}_k)$ is the entropy conservative flux, $\boldsymbol{Q}^i$ satisfies $\boldsymbol{1}^T\boldsymbol{Q}^i = \boldsymbol{1}^T\boldsymbol{B}^i$ and $\boldsymbol{Q}^i\boldsymbol{1} = \boldsymbol{0}$.
- Entropy stable ROM (no hyper-reduction): $\boldsymbol{u} \approx \boldsymbol{V}_N \boldsymbol{u}_N$

$$\boldsymbol{M}_N \frac{\mathrm{d}\boldsymbol{u}_N}{\mathrm{d}t} + \sum_{i=1}^{d} \left(2\boldsymbol{V}_N^T\left(\boldsymbol{Q}^i \circ \boldsymbol{F}^i\right)\boldsymbol{1} + \boldsymbol{V}_b^T \boldsymbol{B}^i \boldsymbol{f}^{i,*}\right) = \boldsymbol{0}, \tag{4}$$

where $\boldsymbol{V}_N$ is the POD basis and $\boldsymbol{V}_b$ is the boundary interpolation matrix.

Entropy stable hyper-reduction

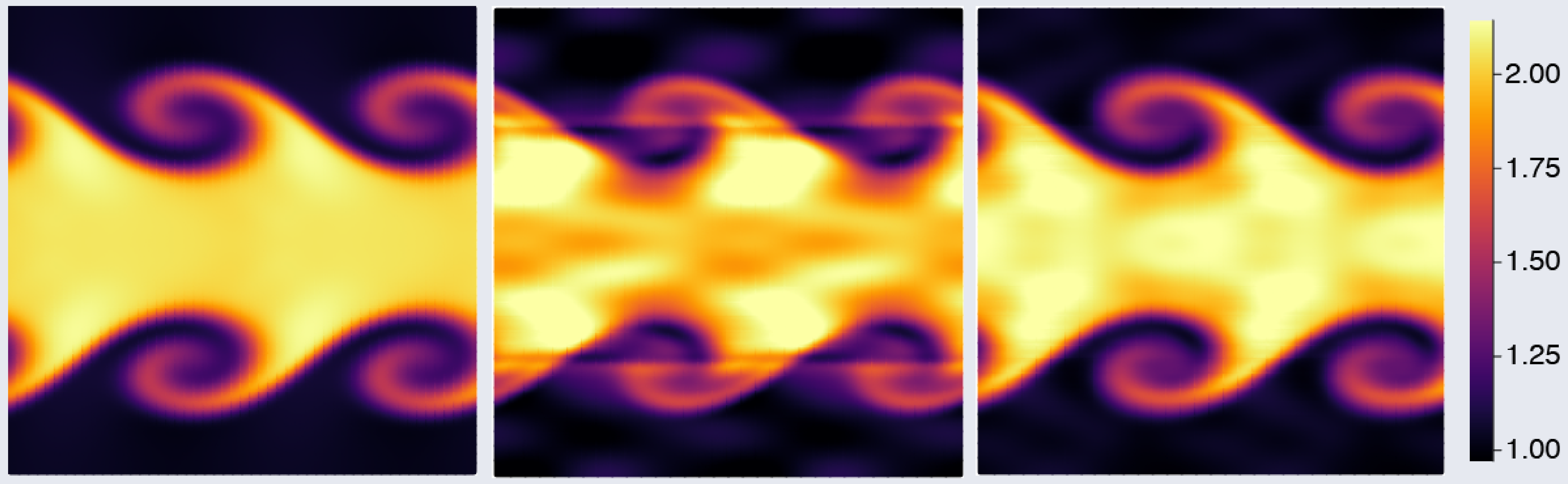
Due to nonlinear terms, the cost of (4) still scales with the dimension of the FOM. We will construct a hyper-reduced ROM from hyper-reduced operators $\overline{\boldsymbol{V}}_N, \overline{\boldsymbol{Q}}^i, \overline{\boldsymbol{V}}_b$, and $\overline{\boldsymbol{B}}^i$:

$$\overline{\boldsymbol{M}}_N \frac{\mathrm{d}\boldsymbol{u}_N}{\mathrm{d}t} + \sum_{i=1}^{d} \left(2\overline{\boldsymbol{V}}_N^T\left(\overline{\boldsymbol{Q}}_N^i \circ \boldsymbol{F}^i\right)\boldsymbol{1} + \overline{\boldsymbol{V}}_b^T \overline{\boldsymbol{B}}^i \boldsymbol{f}^{i,*}\right) = \boldsymbol{0}.$$

where $\overline{\boldsymbol{Q}}_N^i$ is a hybridized SBP operator [2] built from $\overline{\boldsymbol{Q}}^i$. The hyper-reduced ROM is still entropy stable if $\overline{\boldsymbol{Q}}^i$, $\overline{\boldsymbol{V}}_N$ and $\overline{\boldsymbol{B}}^i$, $\overline{\boldsymbol{V}}_b$ satisfy a matrix form of the fundamental theorem of calculus

$$\boldsymbol{1}^T\overline{\boldsymbol{B}}^i\overline{\boldsymbol{V}}_b^i \equiv \boldsymbol{1}^T\overline{\boldsymbol{Q}}^i\overline{\boldsymbol{V}}_N.$$

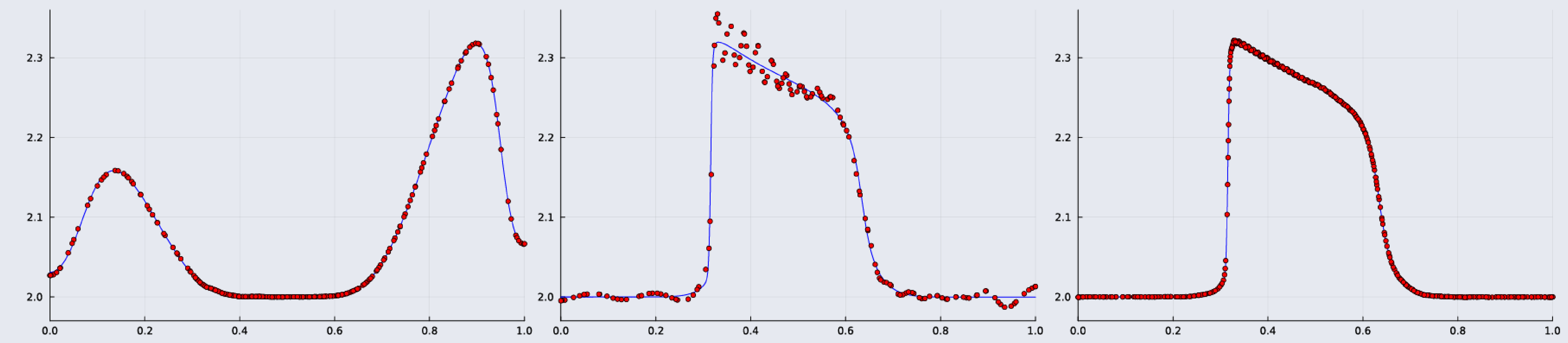
Kelvin-Helmholtz instability (KHI)



(a) FOM (b) 30 modes (c) 50 modes

- Density ρ for the 2D compressible Euler equations at time $T = 3$ on a periodic domain with 32×32 elements and polynomial degree 4 for the DG FOM. We add an artificial viscosity $\epsilon\Delta\boldsymbol{u}$ with $\epsilon = 1e-3$.
- ROMs solutions remain stable independently of solution resolution.

Convergence of ROMs for compressible Euler

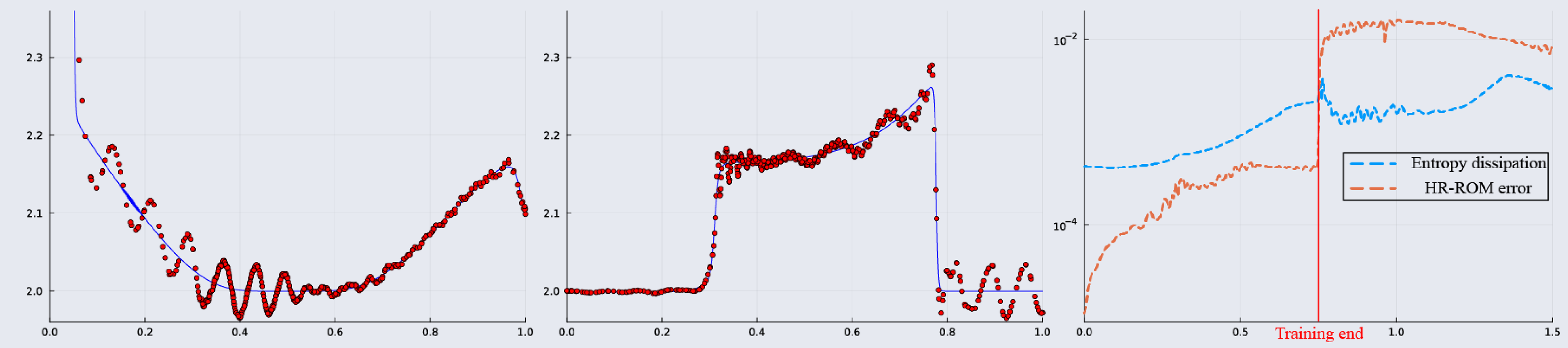


(a) 60 modes, $T = .25$ (b) 40modes, $T = .75$ (c) 100 modes, $T = .75$

Figure: Density ρ. Line: FOM solutions. Red dots: ROM solutions on hyper-reduced nodes.

1D compressible Euler equations with reflective wall boundary conditions, using a FOM with 256 elements, polynomial degree 4, and an artificial viscosity term $\epsilon\Delta\boldsymbol{u}$ with $\epsilon = 2e-4$.

Prediction past training data



(a) 100 modes, $T = 1.0$ (b) 100 modes, $T = 1.5$ (c) Error and entropy dissipation

While the error for predicted solutions past the training window is large, the ROM still remains stable and satisfies an entropy inequality independently of approximation quality.

Volume hyper-reduction

We construct $\overline{\boldsymbol{Q}}^i = \boldsymbol{P}^T\boldsymbol{Q}^i\boldsymbol{P}$, where $\boldsymbol{P}$ is an orthogonal projection with respect to hyper-reduced "quadrature" weights [3] onto some "test" basis [1]. This construction of $\overline{\boldsymbol{Q}}^i$ satisfies

$$\boldsymbol{1}^T\overline{\boldsymbol{Q}}^i\overline{\boldsymbol{V}}_N = \boldsymbol{1}^T\boldsymbol{Q}^i\boldsymbol{V}_N.$$

Boundary hyper-reduction

Goal: find reduced boundary operators $\overline{\boldsymbol{B}}^i, \overline{\boldsymbol{V}}_b^i$ which preserve

$$\boldsymbol{1}^T\overline{\boldsymbol{B}}^i\overline{\boldsymbol{V}}_b^i = \boldsymbol{1}^T\boldsymbol{B}^i\boldsymbol{V}_b^i \quad \left(= \boldsymbol{1}^T\overline{\boldsymbol{Q}}^i\overline{\boldsymbol{V}}_N = \boldsymbol{1}^T\boldsymbol{Q}^i\boldsymbol{V}_N\right). \tag{5}$$

Carathéodory pruning generates an N-point interpolatory positive rule that preserves all moments on N-dimensional space. From

$$\boldsymbol{1}^T\boldsymbol{B}^i\boldsymbol{V}_b = \int \phi_j \boldsymbol{n}^i = \sum_{k=1}^{N_b} \boldsymbol{w}_{b,j}\boldsymbol{n}_j^i\phi_j(\boldsymbol{x}_k), \tag{6}$$

where $N_b \gg N$ is the number of FOM boundary points. Carathéodory pruning generates N reduced boundary nodes with positive weights that can be used to construct $\overline{\boldsymbol{B}}^i, \overline{\boldsymbol{V}}_b^i$.

Conclusion and Acknowledgement

- We present an entropy stable reduced order modeling of nonlinear conservation laws based on high order DG methods.
- We develop structure-preserving hyper-reduction techniques which preserve entropy stability.

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References

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