Entropy stable reduced order modeling of nonlinear conservation laws using discontinuous Galerkin methods

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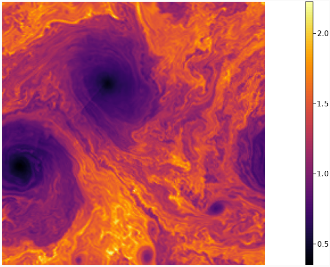
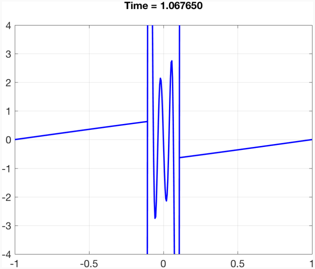
Motivation - entropy stable ROM



(a) Kelvin–Helmholtz instability



(b) 1D Burgers'

- High-fidelity simulations require extreme-scale evaluations.
- Reduced order models (ROMs) enable efficient many-query contexts.
- Both full order models (FOMs) and ROMs tend to blow up around shocks and turbulence.

Chan et al. (2022) *On the entropy projection and the robustness of high order entropy stable discontinuous Galerkin schemes for under-resolved flows.*

Outline

1. Nonlinear conservation laws
2. Literature review
3. FOM and ROM construction on 1D periodic domains
4. FOM and ROM construction on 1D domains with weakly imposed boundary conditions
5. Extension to higher-dimensional domains
6. Discretization of artificial viscosity
7. Numerical experiments

Nonlinear conservation laws

Nonlinear conservation laws and entropy stability

- Energy balance for nonlinear conservation laws (Burgers', shallow water, compressible Euler + Navier-Stokes).

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_i^d \frac{\partial \mathbf{f}^i(\mathbf{u})}{\partial \boldsymbol{x}^i} = 0. \tag{1}$$

- Continuous entropy conservation: convex entropy function $S(\mathbf{u})$, "entropy potential" $\psi(\mathbf{u})$, entropy variables $\mathbf{v}(\mathbf{u})$

$$\int_\Omega \mathbf{v}^T \left(\frac{\partial \mathbf{u}}{\partial t} + \sum_i^d \frac{\partial \mathbf{f}^i(\mathbf{u})}{\partial \boldsymbol{x}^i} \right) = 0, \qquad \boxed{\mathbf{v}(\mathbf{u}) = \frac{\partial S}{\partial \mathbf{u}}}$$

$$\Longrightarrow \int_\Omega \frac{\partial S(\mathbf{u})}{\partial t} + \sum_i^d \left(\mathbf{v}^T \mathbf{f}^i(\mathbf{u}) - \psi^i(\mathbf{u}) \right) \Big|_{-1}^{1} = 0. \tag{2}$$

Nonlinear conservation laws and entropy stability

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Literature review

Literature review - entropy stable numerical methods

- Tadmor introduced second order finite volume methods based on *entropy conservative* numerical fluxes.
- These were extended to high order FVM on periodic domains by Fjordholm, Mishra, and Tadmor.
- Combined with summation by parts (SBP) operators intended for non-periodic boundary conditions, many work further generalized entropy stable discretization to high order discontinuous Galerkin (DG) methods.

Tadmor. (1987) *The numerical viscosity of entropy stable schemes for systems of conservation laws.*

Fjordholm, Mishra, and Tadmor. (2012) *Arbitrarily high-order accurate en- tropy stable essentially nonoscillatory schemes for systems of conservation laws.*

Carpenter et al. (2014) *Entropy stable spectral collocation schemes for the navier–stokes equations: Discontinuous interfaces.*

Gassner et al. (2016) *Shallow water equations: Split-form, entropy stable, well-balanced, and positivity preserving numerical methods.*

Rojas et al. (2021) *On the robustness and performance of entropy stable collocated discontinuous Galerkin methods.*

Literature review - ROM

- Barone and Kalashnikova were the first to combine Galerkin projection and proper orthogonal decomposition (POD) to construct ROMs for linearized compressible flows.
- Carlberg et al. implemented the Gauss–Newton with approximated tensors (GNAT) to stabilize nonlinear reduction.

Barone et al. (2009) *Stable Galerkin reduced order models for linearized compressible flow.*

Kalashnikova and Barone. (2010) *On the stability and convergence of a Galerkin reduced order model (ROM) of compressible flow with solid wall and far-field boundary treatment.*

Carlberg et al. (2013) *The GNAT method for nonlinear model reduction: Effective implementation and application to computational fluid dynamics and turbulent flows.*

Literature review - DG ROM

There's been increasing interests in DG ROMs.

- Du and Yano investigated the benefits of adaptive DG meshes, demonstrating substantial reductions in computational costs while preserving the accuracy of the models.
- Yu and Hesthaven adopted the empirical interpolation method (DEIM) to hyper-reduce upwinding dissipation.

Du and Yano. (2022) *Efficient hyperreduction of high-order discontinuous Galerkin methods: Element-wise and point-wise reduced quadrature formulations.*

Yu and Hesthaven. (2022) *Model order reduction for compressible flows solved using the discontinuous Galerkin methods.*

Literature review - previous work

- However, ROMs in general are not entropy stable and can experience solution instabilities.
- Chan proposed an entropy stable ROMs scheme of nonlinear conservation laws using FVM.
- This thesis therefore proposes a way to construct entropy stable ROMs using high order DG methods, generalizing Chan's previous work.

Chan. (2020) ***Entropy stable reduced order modeling of nonlinear conservation laws.***

FOM and ROM construction on 1D periodic domains

Entropy conservative flux

- The entropy inequality is a generalization of energy stability principle for nonlinear conservation laws.
- The entropy stable schemes that we utilize rely on special numerical fluxes. Denote $\mathbf{u}_L, \mathbf{u}_R$ the left and right solution states. The two-point flux $\mathbf{f}_{EC}(\mathbf{u}_L, \mathbf{u}_R)$ is entropy-conservative if

$$\mathbf{f}_{EC}(\mathbf{u}, \mathbf{u}) = \mathbf{f}(\mathbf{u}), \text{ (consistency)}$$

$$\mathbf{f}_{EC}(\mathbf{u}_L, \mathbf{u}_R) = \mathbf{f}_{EC}(\mathbf{u}_R, \mathbf{u}_L), \text{ (symmetry)}$$

$$(\mathbf{v}_L - \mathbf{v}_R)^T \mathbf{f}_{EC}(\mathbf{u}_L, \mathbf{u}_R) = \psi(\mathbf{u}_L) - \psi(\mathbf{u}_R). \text{ (conservation)}$$

Tadmor. (1987) *The numerical viscosity of entropy stable schemes for systems of conservation laws.*

Tadmor. (2003) *Entropy stability theory for difference approximations of nonlinear conservation laws and related time-dependent problems.*

FOM - local formulation

- We use N_p Gauss-Lobatto points for both interpolation and quadrature (collocation method).

- A local DG formulation on cell D^k is

$$J_k \mathbf{M} \frac{\mathrm{d}\mathbf{u}_k}{\mathrm{dt}} + \left(\left(\mathbf{Q} - \mathbf{Q}^T \right) \circ \mathbf{F}^k \right) \mathbf{1} + \mathbf{B}\mathbf{f}^* = \mathbf{0},$$
$$\mathbf{f}^* = \left[\mathbf{f}_{EC}(\mathbf{u}^+_{1,k}, \mathbf{u}_{1,k}) \quad 0 \quad \cdots \quad 0 \quad \mathbf{f}_{EC}(\mathbf{u}^+_{N_p,k}, \mathbf{u}_{N_p,k}) \right]^T . \tag{3}$$

  For interior cells, $\mathbf{u}^+_{1,k}, \mathbf{u}^+_{N_p,k}$ come from neighbor cells.
  For boundary cells, $\mathbf{u}^+_{1,k}, \mathbf{u}^+_{N_p,k}$ come from the boundary nodes.
- Summation-by-parts (SBP) property: $\mathbf{Q} + \mathbf{Q}^T = \mathbf{B}$.

FOM - global formulation

ROMs are easier to construct based on a global formulation.
Denote global solution vector $\mathbf{u}_\Omega = [\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_K]^T$, then

$$\mathbf{M}_\Omega \frac{\mathrm{d}\mathbf{u}_\Omega}{\mathrm{dt}} + 2(\mathbf{Q}_\Omega \circ \mathbf{F})\mathbf{1} = \mathbf{0},$$

$$\mathbf{M}_\Omega = \mathbf{I} \bigotimes \mathbf{M}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{F}_{11} & \cdots & \mathbf{F}_{1K} \\ \vdots & \ddots & \vdots \\ \mathbf{F}_{K1} & \cdots & \mathbf{F}_{KK} \end{bmatrix},$$

$$\mathbf{Q}_\Omega = \frac{1}{2} \begin{bmatrix} \mathbf{S} & \mathbf{B}_R & & -\mathbf{B}_L \\ -\mathbf{B}_L & \mathbf{S} & \mathbf{B}_R & \\ & -\mathbf{B}_L & \ddots & \mathbf{B}_R \\ \mathbf{B}_R & & -\mathbf{B}_L & \mathbf{S} \end{bmatrix}, \quad \mathbf{S} = (\mathbf{Q} - \mathbf{Q}^T),$$

$$\mathbf{B}_L = \begin{bmatrix} & & 1 \\ & \iddots & \\ 0 & & \end{bmatrix}, \qquad \mathbf{B}_R = \mathbf{B}_L^T = \begin{bmatrix} & & 0 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

Chan and Taylor. (2022) *Efficient computation of Jacobian matrices for entropy stable summation-by-parts schemes.*

FOM - entropy stability

- The global formulation

$$\mathbf{M}_\Omega \frac{\mathrm{d}\mathbf{u}_\Omega}{\mathrm{d}t} + 2(\mathbf{Q}_\Omega \circ \mathbf{F})\mathbf{1} = \mathbf{0}$$

satisfies a semi-discrete entropy conservation condition

$$\mathbf{1}^T \mathbf{M}_\Omega \frac{\mathrm{d}S(\mathbf{u}_\Omega)}{\mathrm{d}t} = 0.$$

For the proof, we utilize $\mathbf{Q}_\Omega$ properties.

- We can show the global differential operator $\mathbf{Q}_\Omega$ is skew-symmetric and has zero row sums

$$\mathbf{Q}_\Omega = -\mathbf{Q}_\Omega^T, \qquad \mathbf{Q}_\Omega \mathbf{1} = \mathbf{0}.$$

ROM - Galerkin projection with POD basis

- Denote $\{\phi_j(\mathbf{x}_i)\}_{j=1}^N$ as our reduced basis and $\mathbf{V}_N$ as the general Vandermonde matrix $(\mathbf{V}_N)_{ij} = \phi_j(\mathbf{x}_i)$.
- Various techniques exist to construct $\mathbf{V}_N$, among which we use proper orthogonal decomposition (POD):

$$\mathbf{V}_{\text{snap}} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T, \qquad \mathbf{V}_N = \mathbf{U}(:, 1:N).$$

- Galerkin projection ROM: assume $\mathbf{u}_\Omega \approx \mathbf{V}_N \mathbf{u}_N$,

$$\mathbf{V}_N^T \mathbf{M}_\Omega \mathbf{V}_N \frac{\mathrm{d}\mathbf{u}_N}{\mathrm{d}t} + 2\mathbf{V}_N^T (\mathbf{Q}_\Omega \circ \mathbf{F})\mathbf{1} = \mathbf{0}. \tag{4}$$

There are two issues: 1) entropy stability proof fails; 2) computational cost is not necessarily reduced.

Liang et al. (2000) *Proper orthogonal decomposition and its applications-Part I: Theory.*

Rowley et al. (2004) *Model reduction for compressible flows using POD and Galerkin projection.*

ROM - entropy stability

- Testing ROM formulation with $\mathbf{v}_N = \mathbf{V}_N^\dagger \mathbf{v}(\mathbf{V}_N \mathbf{u}_N)$,

$$\widetilde{\mathbf{v}}^T (\mathbf{Q} \circ \mathbf{F}) \mathbf{1} = \frac{1}{2} \sum_{ij} \mathbf{Q}_{ij} (\widetilde{\mathbf{v}}_i - \widetilde{\mathbf{v}}_j)^T \mathbf{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j)$$
$$\neq \frac{1}{2} \sum_{ij} \mathbf{Q}_{ij} (\psi(\mathbf{u}_i) - \psi(\mathbf{u}_j)),$$

  where $\widetilde{\mathbf{v}} = \mathbf{V}_N \mathbf{v}_N$.

- Why: $\widetilde{\mathbf{v}}$ are no longer direct mappings from $\mathbf{u}$.
- Solution: we use entropy-projected conservative variables

$$\widetilde{\mathbf{u}} = \mathbf{u}\left(\mathbf{V}_N \mathbf{V}_N^\dagger \mathbf{v}(\mathbf{V}_N \mathbf{u}_N)\right) = \mathbf{u}(\widetilde{\mathbf{v}}).$$

Chan. (2020) *Entropy stable reduced order modeling of nonlinear conservation laws.*

ROM - entropy stability

The final entropy stable ROM is

$$\begin{aligned}
&\mathbf{M}_N \frac{\mathrm{d}\mathbf{u}_N}{\mathrm{dt}} + 2\mathbf{V}_N^T(\mathbf{Q}_\Omega \circ \mathbf{F})\mathbf{1} = \mathbf{0}, \\
&\mathbf{M}_N = \mathbf{V}_N^T \mathbf{M}_\Omega \mathbf{V}_N, \qquad \mathbf{F}_{ij} = \mathbf{f}_{EC}(\widetilde{\mathbf{u}}_i, \widetilde{\mathbf{u}}_j), \\
&\widetilde{\mathbf{u}} = \mathbf{u}\left(\mathbf{V}_N \mathbf{V}_N^\dagger \mathbf{v}\left(\mathbf{V}_N \mathbf{u}_N\right)\right).
\end{aligned} \tag{5}$$

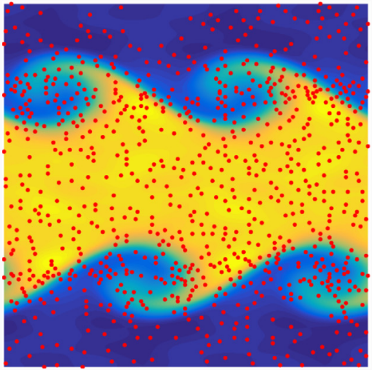
This formulation still has high computational cost!

Hyper-reduction is needed to reduce computational cost.

Hyper-reduction - main idea

- Main idea: weighting and sampling strategy to find indices $\mathcal{I}$ and weights $\mathbf{W}$ such that for nonlinear function g

$$\overline{\mathbf{V}}_N = \mathbf{V}_N(\mathcal{I},:), \qquad \mathbf{V}_N^T g\left(\mathbf{V}_N \mathbf{u}_N\right) \approx \overline{\mathbf{V}}_N^T \mathbf{W} g\left(\overline{\mathbf{V}}_N \mathbf{u}_N\right).$$



Farhat et al. (2015) *Structure-preserving, stability, and accuracy properties of the energy-conserving sampling and weighting method for the hyper reduction of nonlinear finite element dynamic models.*

Two-step hyper-reduction: compression

Hyper-reduction can preserve entropy stability if $\overline{\mathbf{Q}}$ is still skew-symmetric and has zero row sums. To maintain these properties of $\overline{\mathbf{Q}}$, we must apply a two-step hyper-reduction.

First step is compression where we use Galerkin projection with expanded basis approach. Let $\mathbf{V}_t$ be a test basis such that

$$\mathcal{R}(\mathbf{V}_N) \subset \mathcal{R}(\mathbf{V}_t).$$

Then, the intermediate operator (compression) is defined as

$$\widehat{\mathbf{Q}}_t = \mathbf{V}_t^T \mathbf{Q}_\Omega \mathbf{V}_t.$$

Chan. (2020) ***Entropy stable reduced order modeling of nonlinear conservation laws.***

Two-step hyper-reduction: projection

- Second step is projection: we construct test mass matrix

$$\mathbf{M}_t = \mathbf{V}_t(\mathcal{I},:)^T \mathbf{W} \mathbf{V}_t(\mathcal{I},:),$$

then we can construct a projection matrix

$$\mathbf{P}_t = \mathbf{M}_t^{-1} \mathbf{V}_t(\mathcal{I},:)^T \mathbf{W}.$$

Finally, the hyper-reduced differential matrix is defined as

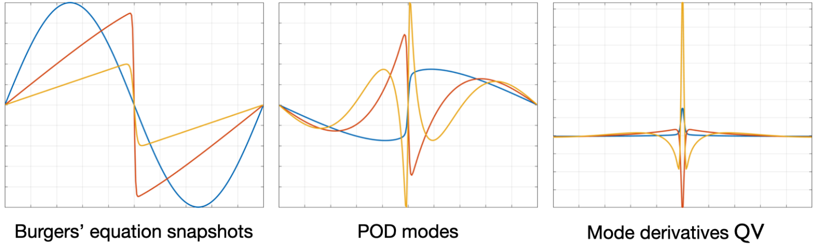
$$\overline{\mathbf{Q}} = \mathbf{P}_t^T \widehat{\mathbf{Q}}_t \mathbf{P}_t = \mathbf{P}_t^T \mathbf{V}_t^T \mathbf{Q}_\Omega \mathbf{V}_t \mathbf{P}_t.$$

- $\overline{\mathbf{Q}}$ is skew-symmetric by construction, and, if $\mathbf{1}$ lies in the span of test basis, $\mathbf{V}_N \mathbf{e} = \mathbf{1}$, $\widehat{\mathbf{Q}}_t \mathbf{e} = \mathbf{0}$ for some coefficient $\mathbf{e}$. Then,

$$\overline{\mathbf{Q}}\mathbf{1} = \mathbf{P}_t^T \widehat{\mathbf{Q}}_t \mathbf{e} = \mathbf{0}.$$

- Thus, the test basis must span $\mathbf{1}$ and $\mathbf{V}_N$. What else?

Two-step hyper-reduction: choice of test basis



- For shocks, POD basis $\mathbf{V}_N$ poorly samples derivative matrix (e.g. $\mathbf{V}_N^T \mathbf{Q}_\Omega \mathbf{V}_N \approx \mathbf{0}$).
- Additionally, the error $\mathbf{Q}_\Omega - \overline{\mathbf{Q}}$ should be orthogonal to $\mathbf{V}_N$ if we use all nodes for hyper-reduction.
- Choice of test basis for DG methods: we augment test basis to also span

$$\boxed{\mathbf{M}_\Omega^{-1} \mathbf{Q}_\Omega^T \mathbf{V}_N.}$$

Two-step hyper-reduction: choice of test basis

Theorem

If $\mathcal{R}(\mathbf{V}_N)$, $\mathcal{R}\left(\mathbf{M}_\Omega^{-1}\mathbf{Q}_\Omega^T\mathbf{V}_N\right) \subset \mathcal{R}(\mathbf{V}_t)$, *then* $\mathbf{V}_N^T\left(\mathbf{Q}_\Omega - \overline{\mathbf{Q}}\right) = \mathbf{0}$ *under "ideal" hyper-reduction (using all nodes).*

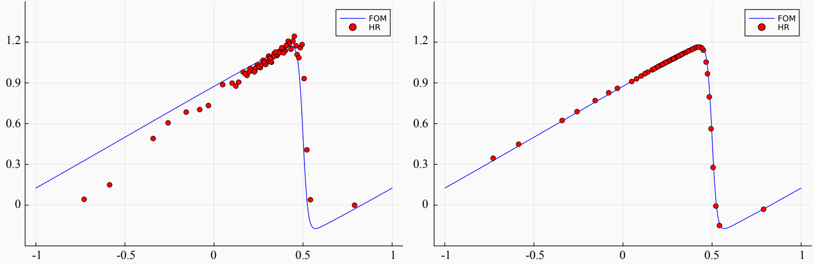


Figure 1: Comparison of including $\mathcal{R}(\mathbf{Q}_\Omega\mathbf{V}_N)$ or $\mathcal{R}(\mathbf{M}_\Omega^{-1}\mathbf{Q}_\Omega^T\mathbf{V}_N)$ in test basis for 1D Burgers' equation using DG methods (25 modes).

Two-step hyper-reduction: algorithm and target space

- Numerous methods exist to select hyper-reduced nodes. We utilize empirical cubature, which chooses hyper-reduced indices and weights greedily by solving

$$\mathbf{V}_{\text{target}}^T \mathbf{w}_{\text{target}} \approx \mathbf{V}_{\text{target}}(\mathcal{I},:)^T \mathbf{w}.$$

- For our DG scheme, we set

$$\mathcal{R}\left(\mathbf{V}_{\text{target}}\right) = \text{span}\left\{\mathbf{V}_N(:,i) \circ \mathbf{V}_N(:,j) \quad \text{for} \quad i,j = 1:N\right\},$$
$$\mathbf{w}_{\text{target}} = \mathbf{V}_{\text{target}}^T \mathbf{J}_\Omega \mathbf{w}_\Omega.$$

- If test mass matrix $\mathbf{V}_t(\mathcal{I},:)^T \mathbf{W} \mathbf{V}_t(\mathcal{I},:)$ is ill-conditioned, we add "stabilizing" points.

Hernádez et al. (2017) ***Dimensional hyper-reduction of nonlinear finite element models via empirical cubature.***

Summary of offline two-step hyper-reduction

Given an N-mode reduced basis $\mathbf{V}_N$, we then

1. Compute a test basis matrix $\mathbf{V}_t$ such that $\mathcal{R}(\mathbf{1}, \mathbf{V}_N, \mathbf{M}_\Omega^{-1}\mathbf{Q}_\Omega^T\mathbf{V}_N) \subset \mathcal{R}(\mathbf{V}_t)$, and compute compressed intermediate operator $\widehat{\mathbf{Q}}_t = \mathbf{V}_t^T\mathbf{Q}\mathbf{V}_t$.
2. Compute a hyper-reduced quadrature using greedy algorithm to obtain a set of hyper-reduced nodes $\mathcal{I}$ and new quadrature weights $\mathbf{w}$, with stabilizing points if necessary to ensure that the test mass matrix is non-singular.
3. Construct the hyper-reduced nodal differentiation matrix $\overline{\mathbf{Q}} = \mathbf{P}_t^T\mathbf{Q}_t\mathbf{P}_t$ using the projection $\mathbf{P}_t = \mathbf{M}_t^{-1}\mathbf{V}_t(\mathcal{I},:)^T\mathbf{W}$ onto the test basis.

Hyper-reduced ROM

The hyper-reduced ROM is

$$
\begin{aligned}
&\overline{\mathbf{M}}_N \frac{\mathrm{d}\mathbf{u}_N}{\mathrm{dt}} + 2\overline{\mathbf{V}}_N^T \left(\overline{\mathbf{Q}} \circ \overline{\mathbf{F}}\right) \mathbf{1} = \mathbf{0} \\
&\overline{\mathbf{V}}_N = \mathbf{V}_N(\mathcal{I},:), \qquad \overline{\mathbf{M}}_N = \overline{\mathbf{V}}_N^T \mathbf{W} \overline{\mathbf{V}}_N, \\
&\mathbf{P} = \overline{\mathbf{M}}_N^{-1} \overline{\mathbf{V}}_N^T \mathbf{W}, \qquad \mathbf{v}_N = \mathbf{P}\mathbf{v}\left(\overline{\mathbf{V}}_N \mathbf{u}_N\right), \\
&\widetilde{\mathbf{v}} = \overline{\mathbf{V}}_N \mathbf{v}_N, \qquad \widetilde{\mathbf{u}} = \mathbf{u}(\widetilde{\mathbf{v}}), \qquad \overline{\mathbf{F}}_{ij} = \mathbf{f}_{EC}(\widetilde{\mathbf{u}}_i, \widetilde{\mathbf{u}}_j),
\end{aligned}
\tag{6}
$$

which semi-discretely conserves the sampled and weighted average entropy

$$
\mathbf{1}^T \mathbf{W} \frac{\mathrm{d}S\left(\overline{\mathbf{V}}_N \mathbf{u}_N\right)}{\mathrm{dt}} = 0.
$$

FOM and ROM construction on 1D domains with weakly imposed boundary conditions

FOM - weakly imposed BC

From local formulation

$$J_k \mathbf{M} \frac{\mathrm{d}\mathbf{u}_k}{\mathrm{dt}} + \left(\left(\mathbf{Q} - \mathbf{Q}^T\right) \circ \mathbf{F}^k\right)\mathbf{1} + \mathbf{B}\mathbf{f}^* = \mathbf{0},$$

we can construct a global formulation

$$\mathbf{M}_\Omega \frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} + 2(\mathbf{Q}_\Omega \circ \mathbf{F})\mathbf{1} + \mathbf{B}_\Omega \mathbf{f}_\Omega^* = \mathbf{0},$$

$$\mathbf{Q}_\Omega = \frac{1}{2}\begin{bmatrix} \mathbf{S} & \mathbf{B}_R & & \\ -\mathbf{B}_L & \mathbf{S} & \mathbf{B}_R & \\ & -\mathbf{B}_L & \ddots & \mathbf{B}_R \\ & & -\mathbf{B}_L & \mathbf{S} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

where $\mathbf{Q}_\Omega$ satisfies global SBP property $\mathbf{Q}_\Omega + \mathbf{Q}_\Omega^T = \mathbf{B}_\Omega$.
Under proper boundary conditions, this still conserves entropy.

Chen and Shu. (2017) *Entropy stable high order discontinuous Galerkin methods with suitable quadrature rules for hyperbolic conservation laws.*

ROM - Hybridized operator

1D hyper-reduced operator $\overline{\mathbf{Q}}$ satisfies a generalized SBP property

$$\overline{\mathbf{Q}} + \overline{\mathbf{Q}}^T = \mathbf{E}^T \mathbf{B} \mathbf{E},$$

$$\mathbf{B} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \qquad \mathbf{V}_{bt} = \begin{bmatrix} \mathbf{V}_t(1,:) \\ \mathbf{V}_t(\text{end},:) \end{bmatrix}, \qquad \mathbf{E} = \mathbf{V}_{bt} \mathbf{P}_t.$$

To impose nonlinear boundary conditions, we employ a hybridzed SBP operator

$$\overline{\mathbf{Q}}_h = \frac{1}{2} \begin{bmatrix} \overline{\mathbf{Q}} - \overline{\mathbf{Q}}^T & \mathbf{E}^T \mathbf{B} \\ -\mathbf{B}\mathbf{E} & \mathbf{B} \end{bmatrix},$$

satisfying a block SBP property

$$\overline{\mathbf{Q}}_h + \overline{\mathbf{Q}}_h^T = \begin{bmatrix} \mathbf{0} & \\ & \mathbf{B} \end{bmatrix} = \mathbf{B}_h.$$

Chan. (2018) *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Hyper-reduced ROM - weakly imposed BC

One can show

$$\overline{\mathbf{Q}}_h \mathbf{1} = \frac{1}{2}\begin{bmatrix} \overline{\mathbf{Q}}\mathbf{1} - \overline{\mathbf{Q}}^T\mathbf{1} + \mathbf{E}^T\mathbf{B}\mathbf{1} \\ -\mathbf{B}\mathbf{E}\mathbf{1} + \mathbf{B}\mathbf{1} \end{bmatrix} = \mathbf{0}.$$

Denote

$$\mathbf{V}_b = \begin{bmatrix} \mathbf{V}_N(1,:) \\ \mathbf{V}_N(\text{end},:) \end{bmatrix}, \qquad \overline{\mathbf{V}}_h = \begin{bmatrix} \overline{\mathbf{V}}_N \\ \mathbf{V}_b \end{bmatrix}.$$

Then we can build the following hyper-reduced ROM

$$\overline{\mathbf{M}}_N \frac{\mathrm{d}\mathbf{u}_N}{\mathrm{d}t} + 2\overline{\mathbf{V}}_h^T(\overline{\mathbf{Q}}_h \circ \overline{\mathbf{F}})\mathbf{1} + \mathbf{V}_b^T\mathbf{B}(\mathbf{f}_b^* - \mathbf{f}(\widetilde{\mathbf{u}}_b)) = \mathbf{0}$$

$$\overline{\mathbf{V}}_N = \mathbf{V}_N(\mathcal{I},:), \qquad \overline{\mathbf{M}}_N = \overline{\mathbf{V}}_N^T\mathbf{W}\overline{\mathbf{V}}_N,, \qquad \mathbf{P} = \overline{\mathbf{M}}_N^{-1}\overline{\mathbf{V}}_N^T\mathbf{W},$$

$$\mathbf{v}_N = \mathbf{P}\mathbf{v}\left(\overline{\mathbf{V}}_N\mathbf{u}_N\right), \qquad \widetilde{\mathbf{v}} = \overline{\mathbf{V}}_h\mathbf{v}_N, \qquad \widetilde{\mathbf{v}}_b = \mathbf{V}_b\mathbf{v}_N,$$

$$\widetilde{\mathbf{u}} = \mathbf{u}(\widetilde{\mathbf{v}}), \qquad \widetilde{\mathbf{u}}_b = \mathbf{u}(\widetilde{\mathbf{v}}_b), \qquad \overline{\mathbf{F}}_{ij} = \mathbf{f}_{EC}(\widetilde{\mathbf{u}}_i, \widetilde{\mathbf{u}}_j).$$

Hyper-reduced ROM - weakly imposed BC

By the block SBP property of $\overline{\mathbf{Q}}_h$, previous hyper-reduced ROM formulation is equivalent to

$$\overline{\mathbf{M}}_N \frac{\mathrm{d}\mathbf{u}_N}{\mathrm{dt}} + \overline{\mathbf{V}}_h^T \left(\left(\overline{\mathbf{Q}}_h - \overline{\mathbf{Q}}_h^T \right) \circ \overline{\mathbf{F}} \right) \mathbf{1} + \mathbf{V}_b^T \mathbf{B} \mathbf{f}^* = \mathbf{0}, \tag{7}$$

which admits entropy conservation for entropy conservative boundary flux

$$\mathbf{1}^T \mathbf{W} \frac{\mathrm{d}S(\mathbf{V}_N \mathbf{u}_N)}{\mathrm{dt}} = 0.$$

Extension to higher-dimensional domains

FOM - global formulation

For a periodic domain with dimension d,

$$\mathbf{M}_\Omega \frac{\mathrm{d}\mathbf{u}_\Omega}{\mathrm{d}t} + \sum_{i=1}^{d} 2\left(\mathbf{Q}_\Omega^i \circ \mathbf{F}^i\right)\mathbf{1} = \mathbf{0}, \quad (8)$$

Explicit form are complicated, but for vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{Q}_\Omega^i$ satisfies

$$\mathbf{v}^T \mathbf{Q}_\Omega^i \mathbf{u} = \sum_k \left(\int_{D^k} \frac{\partial u}{\partial x^i} v + \int_{\partial D^k} \frac{1}{2} [\![u]\!] n^i v \right). \quad (9)$$

To follow similar entropy stability proof, we show

$$\mathbf{v}^T \mathbf{Q}_\Omega^i \mathbf{u} = -\mathbf{u}^T \mathbf{Q}_\Omega^i \mathbf{v}, \qquad \text{(skew-symmetry)}$$

and $\mathbf{Q}_\Omega^i$ has zero row sums.

Hyper-reduced ROM - dimension by dimension

Applying hyper-reduction dimension by dimension

$$\overline{\mathbf{Q}}^i = (\mathbf{P}_t^i)^T (\mathbf{V}_t^i)^T \mathbf{Q}_\Omega^i \mathbf{V}_t^i \mathbf{P}_t^i,$$

we get the entropy conservative hyper-reduced ROM

$$\begin{aligned}
&\overline{\mathbf{M}}_N \frac{\mathrm{d}\mathbf{u}_N}{\mathrm{dt}} + \sum_{i=1}^{d} \left(2\overline{\mathbf{V}}_N^T \left(\overline{\mathbf{Q}}^i \circ \overline{\mathbf{F}}^i \right) \mathbf{1} \right) = \mathbf{0} \\
&\overline{\mathbf{V}}_N = \mathbf{V}_N(\mathcal{I}, :), \qquad \overline{\mathbf{M}}_N = \overline{\mathbf{V}}_N^T \mathbf{W} \overline{\mathbf{V}}_N, \\
&\mathbf{P} = \overline{\mathbf{M}}_N^{-1} \overline{\mathbf{V}}_N^T \mathbf{W}, \qquad \mathbf{v}_N = \mathbf{P}\mathbf{v}\left(\overline{\mathbf{V}}_N \mathbf{u}_N \right), \\
&\widetilde{\mathbf{v}} = \overline{\mathbf{V}}_N \mathbf{v}_N, \qquad \widetilde{\mathbf{u}} = \mathbf{u}(\widetilde{\mathbf{v}}), \qquad \overline{\mathbf{F}}_{jk}^i = \mathbf{f}_{EC}^i(\widetilde{\mathbf{u}}_j, \widetilde{\mathbf{u}}_k).
\end{aligned} \tag{10}$$

FOM - weakly imposed BC

We can extend 1D formulation

$$\mathbf{M}_\Omega \frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} + 2(\mathbf{Q}_\Omega \circ \mathbf{F})\mathbf{1} + \mathbf{B}_\Omega \mathbf{f}^* = \mathbf{0}$$

to higher dimensions:

$$\mathbf{M}_\Omega \frac{\mathrm{d}\mathbf{u}_\Omega}{\mathrm{dt}} + \sum_{i=1}^{d} \left(2(\mathbf{Q}_\Omega^i \circ \mathbf{F}^i)\mathbf{1} + \mathbf{B}_\Omega^i \mathbf{f}^{i,*} \right) = \mathbf{0}. \tag{11}$$

We notice that the hyper-reduction on volume and boundary flux terms are independent. We can use previous approach to hyper-reduce volume terms, but must now introduce boundary hyper-reduction.

Hyper-reduced ROM - weakly imposed BC

We first derive conditions for boundary hyper-reduction $(\mathcal{I}_b, \mathbf{w}_b)$ which guarantee entropy stability. Denote

$$\overline{\mathbf{B}}^i = \operatorname{diag}\left(\mathbf{n}^i\right)\mathbf{W}_b, \qquad \overline{\mathbf{V}}_{bt} = \mathbf{V}_{bt}\left(\mathcal{I}_b, :\right), \qquad \overline{\mathbf{E}}^i = \overline{\mathbf{V}}_{bt}^i \mathbf{P}_t^i.$$

The hybridized SBP operator along the ith coordinate is

$$\overline{\mathbf{Q}}_h^i = \begin{bmatrix} \overline{\mathbf{Q}}^i - \left(\overline{\mathbf{Q}}^i\right)^T & \left(\overline{\mathbf{E}}^i\right)^T \overline{\mathbf{B}}^i \\ -\overline{\mathbf{B}}^i \overline{\mathbf{E}}^i & \overline{\mathbf{B}}^i \end{bmatrix}.$$

Hyper-reduced ROM:

$$\overline{\mathbf{M}}_N \frac{\mathrm{d}\mathbf{u}_N}{\mathrm{dt}} + \sum_{i=1}^{d} \left(\overline{\mathbf{V}}_h^T \left(\left(\overline{\mathbf{Q}}_h^i - \left(\overline{\mathbf{Q}}_h^i\right)^T \right) \circ \overline{\mathbf{F}}^i \right) \mathbf{1} + \overline{\mathbf{V}}_b^T \overline{\mathbf{B}}^i \mathbf{f}^{i,*} \right) = \mathbf{0}. \quad (12)$$

Boundary hyper-reduction

To preserve entropy stability, we need to ensure that $\overline{\mathbf{Q}}_h^i \mathbf{1} = \mathbf{0}$. In general,

$$\overline{\mathbf{Q}}_h^i \mathbf{1} = \begin{bmatrix} -\left(\overline{\mathbf{Q}}^i\right)^T \mathbf{1} + \left(\overline{\mathbf{E}}^i\right)^T \overline{\mathbf{B}}^i \mathbf{1} \\ \mathbf{0} \end{bmatrix}.$$

Note that

$$\left(\overline{\mathbf{Q}}^i\right)^T \mathbf{1} = \left(\overline{\mathbf{E}}^i\right)^T \overline{\mathbf{B}}^i \mathbf{1} \iff \boxed{\mathbf{1}^T \mathbf{Q}_\Omega^i \mathbf{V}_t^i = \mathbf{1}^T \overline{\mathbf{B}}^i \overline{\mathbf{V}}_{bt}^i}.$$

We need to enforce this equality (matrix form of the fundamental theorem of calculus) for boundary hyper-reduction.

Boundary hyper-reduction - Carathéodory pruning

Carathéodory's theorem states that, for any M-point positive quadrature rule exact on N-dimensional space span$\{v_1, ..., v_N\}$, we can always generate a new N-point interpolatory positive rule which preserves all moments.

Boundary hyper-reduction - Carathéodory pruning

In matrix form,

$$\mathbf{V}^T\mathbf{w} = \begin{bmatrix} v_1(x_1) & v_1(x_2) & \cdots & v_1(x_M) \\ v_2(x_1) & v_2(x_2) & \cdots & v_2(x_M) \\ \vdots & \vdots & \ddots & \vdots \\ v_N(x_1) & v_N(x_2) & \cdots & v_N(x_M) \end{bmatrix} \begin{bmatrix} w_1 \\ w_1 \\ \vdots \\ w_M \end{bmatrix} = \mathbf{m}.$$

Carathéodory's theorem then states that $\mathbf{m}$ lies in the convex hull of a subset of N columns of $\mathbf{V}^T$.

In our case,

$$\mathbf{1}^T\mathbf{B}^i\mathbf{V}^i_{bt} = \int \phi^i_{bt,j}(\mathbf{x}_k)\mathbf{n}^i = \sum_{k=1} \mathbf{w}_{b,j}\mathbf{n}^i_j\phi_{bt,j}(\mathbf{x}_k).$$

We can concatenate all dimensions such that

$$\mathbf{V} = \left[\text{diag}\left(\mathbf{n}^1\right)\mathbf{V}^1_{bt} \quad \cdots \quad \text{diag}\left(\mathbf{n}^d\right)\mathbf{V}^d_{bt}\right]^T.$$

Van Den Bos, Sanderse, Bierbooms, Van Bussel (2020). *Generating nested quadrature rules with positive weights based on arbitrary sample sets.*

Discretization of artificial viscosity

Discretization of artificial viscosity

To remove spurious oscillations, we add artificial viscosity

$$\epsilon \Delta \mathbf{u}$$

to the system. We utilize the Bassi-Rebay-1 (BR-1) scheme to approximate the Laplacian. On each element D_k, we define $\sigma \approx \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ as

$$\sigma = (J_k \mathbf{M})^{-1} \left(\mathbf{Q}\mathbf{u} + \mathbf{V}_f^T \mathbf{u}_{\mathsf{flux}} \right), \qquad \Delta \mathbf{u} \approx \mathbf{Q}\sigma + \mathbf{V}_f^T \sigma_{\mathsf{flux}},$$

where $\mathbf{u}_{\mathsf{flux}}$ and σ_{flux} are defined by using the central flux.

Manzanero et al. (2018). *The bassi rebay 1 scheme is a special case of the symmetric interior penalty formulation for discontinuous Galerkin discretisations with gauss–lobatto points.*

Discretization artificial viscosity

- Since the interactions at the element faces are already accounted for our global differentiation matrix $\mathbf{Q}_\Omega$, implementing the BR-1 scheme is straightforward by applying the global operator $\mathbf{Q}_\Omega$ twice

  $$\sigma_\Omega = \mathbf{M}_\Omega^{-1}\mathbf{Q}_\Omega\mathbf{u}_\Omega, \qquad \Delta\mathbf{u}_\Omega \approx \mathbf{Q}_\Omega\sigma_\Omega.$$

  For ROMs, we simply use Galerkin projection without hyper-reduction. For high-dimensional domains, we sum up viscosity terms over each dimension.
- We cannot prove entropy stability of the viscous term even for FOMs, but we observe entropy dissipation for all numerical experiments.

Numerical experiments

FVM and DG ROMs comparison

- We first compare the ROM performance between FVM and DG methods by examining the 1D compressible Euler equations over periodic domain $[-1, 1]$.
- The initial condition is an isentropic Gaussian wave such that

$$\rho = 1 + 0.1e^{-25x^2}, \qquad u = 0.1\sin(\pi x), \qquad p = \rho^\gamma.$$

  We add artificial viscosity with coefficient 5×10^{-4}.
- The FOM dim is fixed at 1,024. Simulations are run until two distinct final times $T = 0.1, 1.0$.

FVM and DG ROMs comparison

Error/HR nodes	$p = 0$ (FVM)	$p = 3$	$p = 7$
$N = 10$	3.04e-5/ 28	2.98e-5/ 32	3.04e-5/ 28
$N = 15$	2.14e-7/ 45	3.65e-7/ 75	2.21e-7/ 45
$N = 20$	5.08e-9/ 77	6.75e-8/ 139	6.00e-9/ 90

Table 1: 1D compressible Euler, $T = 0.1$

Smooth solution at $T = 0.1$. Both schemes have similar accuracy, while DG yields more nodes for certain interpolation degree and number of modes.

FVM and DG ROMs comparison

Error/HR nodes	$p = 0$ (FVM)	$p = 3$	$p = 7$
$N = 20$	2.57e-5/ 48	2.57e-5/ 66	2.57e-5/ 48
$N = 30$	1.34e-6/ 74	1.30e-6/ 131	1.37e-6/ 74
$N = 40$	2.83e-8/ 101	3.65e-8/ 284	3.36e-8/ 140

Table 2: 1D compressible Euler, $T = 1.0$

System now exhibits a shock at $T = 1.0$. Both schemes still achieve similar accuracy, but DG still yields more nodes especially for $p = 3$.

DG ROM Example 1 - 1D reflective wall boundary conditions

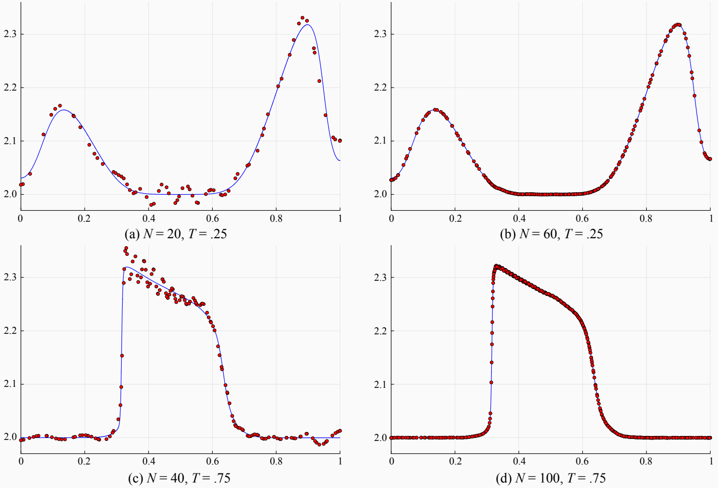
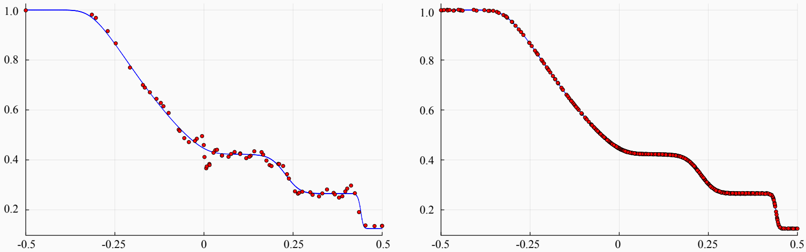


Figure 2: 1D Compressible Euler (reflective wall). FOM dim: 2048. Viscosity: 2×10^{-4}. Runtime $T = .75$.

DG ROM Example 2 - Sod shock tube



(a) $N = 20$, $T = .25$ (b) $N = 100$, $T = .25$

Figure 3: FOM dim: 2048. Viscosity: 5×10^{-4}. Runtime T = $.25$.

DG ROM Example 3 - Kelvin-Helmholtz instability

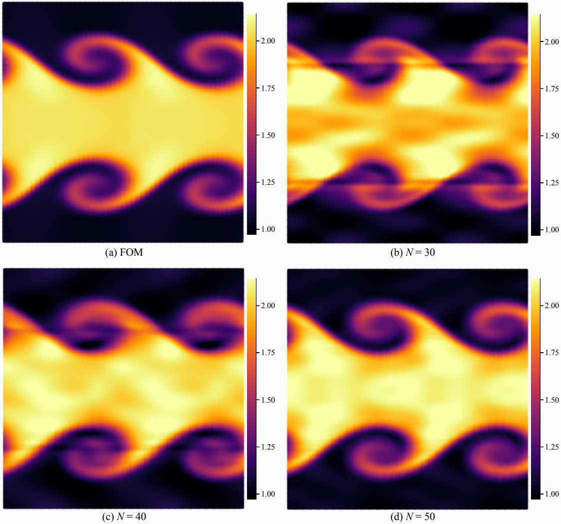


Figure 4: FOM dim: 25,600. Viscosity: 1×10^{-3}. Runtime $T = 3.0$.

DG ROM Example 4 - 2D reflective wall boundary conditions

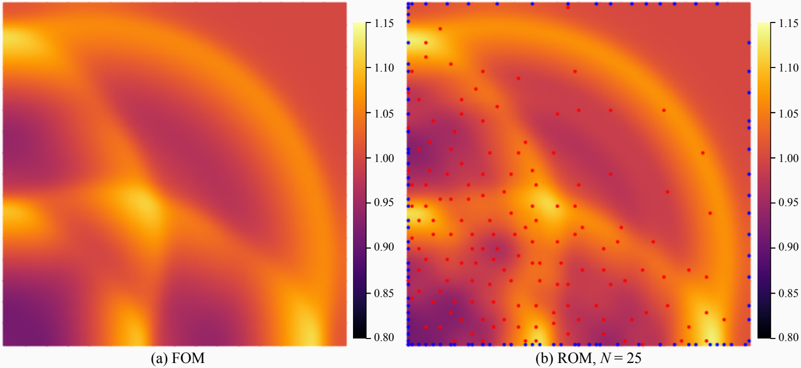


Figure 5: 2D compressible Euler (reflective wall). FOM dim: 6400. Viscosity: 1×10^{-3}. Run time $T = 1.0$. Boundary hyper-reduced by Carathéodory pruning (blue nodes).

Convective entropy contribution

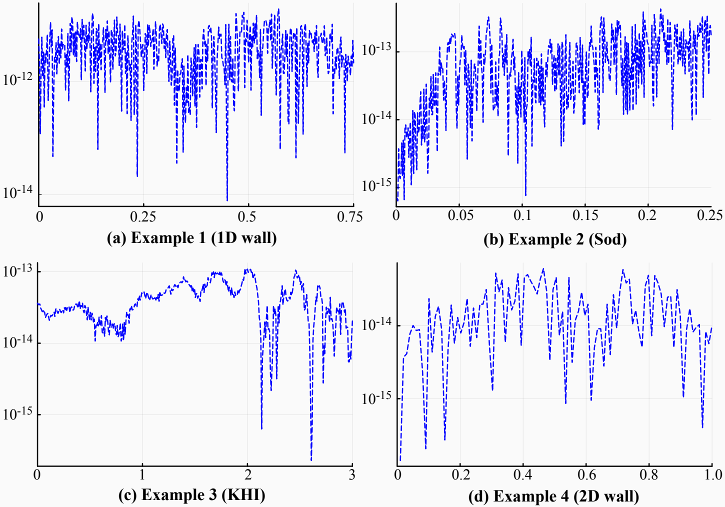


Figure 6: Convective entropy $|\mathbf{v}_N^T \overline{\mathbf{V}}_h^T (\overline{\mathbf{Q}}_h \circ \mathbf{F}) \mathbf{1}|$ of DG ROMs.

Summary and future work

Summary

In this work, we present an entropy stable reduced order modeling of nonlinear conservation laws based on high order DG methods, in which we develop structure-preserving hyper-reduction techniques to preserve entropy stability. Specifically,

- we incorporate weighted test basis for volume hyper-reduction accuracy in our DG scheme;
- we generalize dimension-by-dimension hyper-reduction;
- we utilize Carathéodory pruning for boundary hyper-reduction.

Future work - interface flux dissipation

For future work, we plan to

- incorporate interface flux dissipation by developing entropy stable hyper-reduction of dissipation terms.

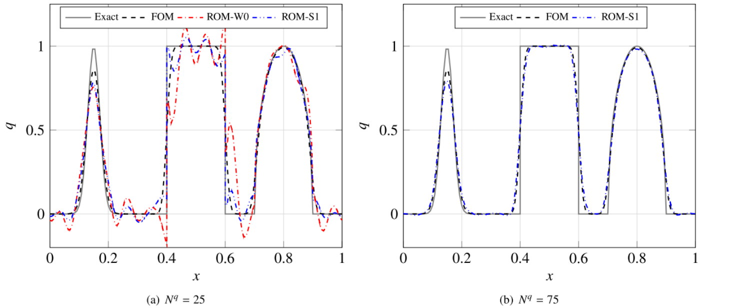


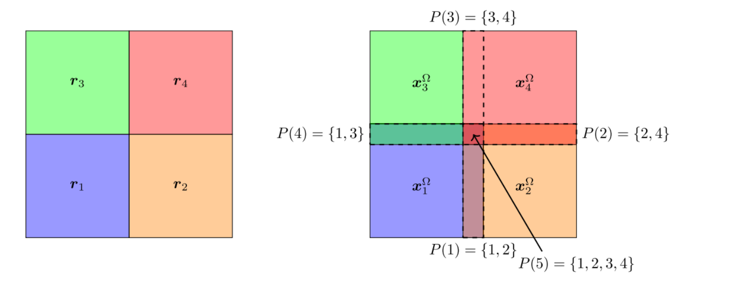
Fig. 2. Solution of the one-dimensional linear convection problem at $t = 1$. $P2$. For this case, the ROM-W0 method blows up at $N^q = 75$.

Yu and Hesthaven. (2022) *Model order reduction for compressible flows solved using the discontinuous Galerkin methods.*

Future work - domain decomposition

For future work, we plan to

- implement domain decomposition (DD) to enhance accuracy and efficiency and extend numerical experiments to compressible Navier-Stokes equations.



Hoang, Choi, and Carlberg. (2021) *Domain-decomposition least-squares Petrov–Galerkin (DD-LSPG) nonlinear model reduction.*

Diaz, Choi, and Heinkenschloss. (2024) *A fast and accurate domain decomposition nonlinear manifold reduced order model.*