Entropy stable reduced order modeling of nonlinear conservation laws using discontinuous Galerkin methods

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Abstract

- Extension of entropy stable reduced order models (ROMs) of nonlinear conservation laws from finite volume methods [1] to high order discontinuous Galerkin (DG) methods.
- Hyper-reduction techniques: gappy proper orthogonal decomposition (gappy-POD) and Carathéodory pruning.

Background

- Nonlinear conservation laws with conservative variables $\boldsymbol{u} \in \mathbb{R}^n$ on domain Ω

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{i=1}^{d} \frac{\partial \boldsymbol{f}_i(\boldsymbol{u})}{\partial \boldsymbol{x}_i} = 0, \qquad (\boldsymbol{x}, t) \in \Omega \times [0, \infty). \tag{1}$$

- Many systems admit an entropy inequality with convex entropy function $S(\boldsymbol{u})$

$$\int_{\Omega} \frac{\partial S(\boldsymbol{u})}{\partial t} d\boldsymbol{x} + \sum_{i=1}^{d} \int_{\partial\Omega} (\boldsymbol{v}^T \boldsymbol{f}_i(\boldsymbol{u}) - \psi_i(\boldsymbol{u}))\boldsymbol{n}^i \leq 0. \tag{2}$$

- Entropy stability is a generalization of energy stability.

Reduced order modeling

- The global DG formulation of (1) is

$$\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{dt}} + \sum_{i=1}^{d} (2(\boldsymbol{Q}^i \circ \boldsymbol{F}^i)\boldsymbol{1} + \boldsymbol{B}^i \boldsymbol{f}^{i,\star}) = \boldsymbol{0}, \tag{3}$$

  where $(\boldsymbol{F}^i)_{j,k} = \boldsymbol{f}_i(\boldsymbol{u}_j, \boldsymbol{u}_k)$ is the entropy conservative flux, $\boldsymbol{Q}^i$ is a summation by parts (SBP) operator with $\boldsymbol{Q}^i\boldsymbol{1} = 0$.
- Galerkin projection ROM ($\boldsymbol{V}_N$ is the POD basis):

$$\boldsymbol{M}_N\frac{\mathrm{d}\boldsymbol{u}_N}{\mathrm{dt}} + \sum_{i=1}^{d} (2\boldsymbol{V}_N^T(\boldsymbol{Q}^i \circ \boldsymbol{F}^i)\boldsymbol{1} + \boldsymbol{V}_b^T\boldsymbol{B}^i \boldsymbol{f}^{i,\star}) = \boldsymbol{0}, \tag{4}$$

  $\boldsymbol{u} \approx \boldsymbol{V}_N\boldsymbol{u}_N$ and $\boldsymbol{V}_b$ is a boundary submatrix of $\boldsymbol{V}_N$.
- Due to nonlinear terms, the cost of (4) still scales with the dimension of the FOM. We will construct a hyper-reduced ROM from hyper-reduced operators $\overline{\boldsymbol{V}}_N, \overline{\boldsymbol{Q}}^i, \overline{\boldsymbol{V}}_b$, and $\overline{\boldsymbol{B}}^i$:

$$\overline{\boldsymbol{M}}_N\frac{\mathrm{du_N}}{\mathrm{dt}} + \sum_{i=1}^{d} (\overline{\boldsymbol{V}}_N^T((\overline{\boldsymbol{Q}}^i - \overline{\boldsymbol{Q}}^{i,T}) \circ \boldsymbol{F}^i)\boldsymbol{1} + \overline{\boldsymbol{V}}_b^T \overline{\boldsymbol{B}}^i \boldsymbol{f}^{i,\star}) = \boldsymbol{0}.$$

Hyper-reduction of volume terms

First, we utilize a greedy algorithm [3] to construct hyper-reduced indices I and weights $\boldsymbol{w}$ for a target space

$$\boldsymbol{V}_{\text{target}}^T \boldsymbol{w}_{\text{target}} \approx \boldsymbol{V}_{\text{target}}(I,:)^T \boldsymbol{w}, \qquad \boldsymbol{w}_{\text{target}}, \boldsymbol{w} > 0, \qquad \overline{\boldsymbol{M}}_N = \boldsymbol{V}_N(I,:)^T \text{diag}(\boldsymbol{w}) \boldsymbol{V}_N(I,:). \tag{5}$$

Then, we use a two-step "compress and project" procedure to build $\boldsymbol{Q}_t^i$, starting with a test basis $\boldsymbol{V}_t^i$ such that $\boldsymbol{1}$, $\boldsymbol{V}_N$, and $\boldsymbol{Q}^i\boldsymbol{V}_N$ are in its range

$$\widehat{\boldsymbol{Q}}_t^i = (\boldsymbol{V}_t^i)^T \boldsymbol{Q}^i \boldsymbol{V}_t^i, \qquad \overline{\boldsymbol{V}}_t^i = \boldsymbol{V}_t^i(I,:), \qquad \boldsymbol{Q}_t^i = ((\overline{\boldsymbol{V}}_t^i)^\dagger)^T \widehat{\boldsymbol{Q}}_t^i (\overline{\boldsymbol{V}}_t^i)^\dagger \quad \text{(gappy-POD)}. \tag{6}$$

$\overline{\boldsymbol{Q}}^i$ is the hybridized SBP differentiation operator [2] along the ith coordinate

$$\overline{\boldsymbol{Q}}^i = \frac{1}{2}\begin{bmatrix} \boldsymbol{Q}_t^i - (\boldsymbol{Q}_t^i)^T & \overline{\boldsymbol{E}}_i^T \overline{\boldsymbol{B}}^i \\ -\overline{\boldsymbol{B}}^i \overline{\boldsymbol{E}}_i & \overline{\boldsymbol{B}}^i \end{bmatrix}. \tag{7}$$

Hyper-reduction of boundary terms using Carathéodory pruning

Define $\boldsymbol{E}^i = \boldsymbol{V}_{bt}^i \boldsymbol{P}_t^i$, where $\boldsymbol{V}_{bt}^i$ is a boundary submatrix of $\boldsymbol{V}_t^i$ and $\boldsymbol{B}^i = \text{diag}(\boldsymbol{n}^i)\text{diag}(\boldsymbol{w}_b)$. Our goal is to find hyper-reduced boundary matrix $\overline{\boldsymbol{B}}^i$ such that

$$\boldsymbol{1}^T \overline{\boldsymbol{B}}^i \overline{\boldsymbol{E}}^i = \boldsymbol{1}^T \boldsymbol{B}^i \boldsymbol{E}^i = \boldsymbol{1}^T \boldsymbol{Q}_t^i. \tag{8}$$

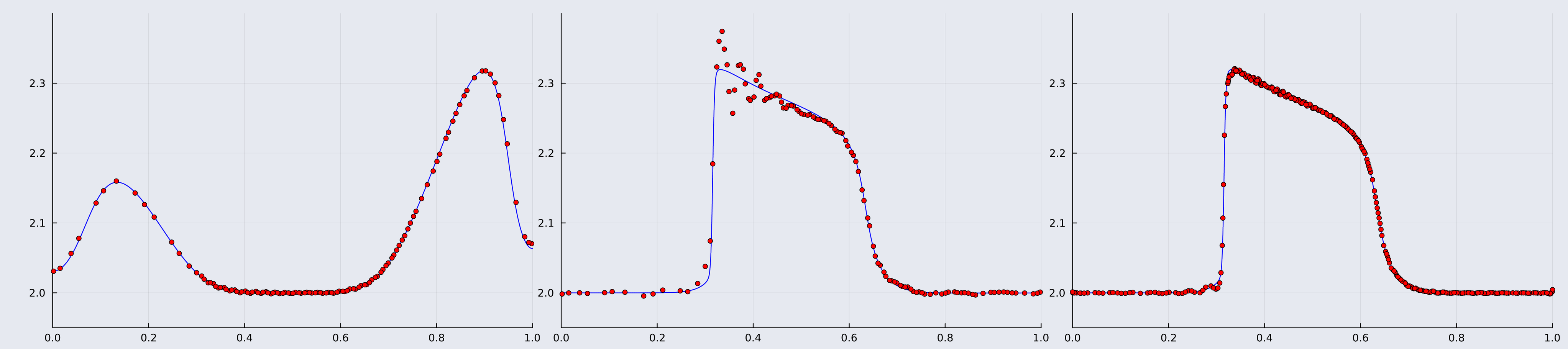
Carathéodory's Theorem states that, given any positive quadrature rule on a space V with $\text{dim}(V) = N$, we can generate an N-point interpolatory positive rule that preserves all moments. Therefore, from

$$\boldsymbol{1}^T \boldsymbol{B}^i \boldsymbol{V}_{bt} = \int \phi_{t,j} \boldsymbol{n}^i = \sum_k \boldsymbol{w}_{b,j} \boldsymbol{n}_j^i \phi_{t,j}(\boldsymbol{x}_k), \tag{9}$$

we are able to select N boundary nodes I_b with new positive weights $\overline{\boldsymbol{w}}_b$ from it to construct

$$\overline{\boldsymbol{V}}_{bt}^i = \boldsymbol{V}_{bt}^i(I_b,:), \qquad \overline{\boldsymbol{E}}_i = \overline{\boldsymbol{V}}_{bt}^i \boldsymbol{P}_t^i, \qquad \overline{\boldsymbol{B}}^i = \text{diag}(\boldsymbol{n}^i)\text{diag}(\overline{\boldsymbol{w}}_b), \qquad \boldsymbol{1}^T \overline{\boldsymbol{B}}^i \overline{\boldsymbol{E}}^i = \boldsymbol{1}^T \boldsymbol{B}^i \boldsymbol{E}^i. \tag{10}$$

Numerical experiments



(a) 60 modes, $T = .25$ (b) 60 modes, $T = .75$ (c) 120 modes, $T = .75$

Figure: Density ρ. FOM solutions are displayed using line plots, while ROM solutions are indicated on hyper-reduced nodes in red.

1D compressible Euler equations with reflective wall boundary conditions, using a FOM with 256 elements, polynomial degree $N = 4$, and an artificial viscosity term $\epsilon\Delta\boldsymbol{u}$ with $\epsilon = 2e-4$.

Numerical experiments

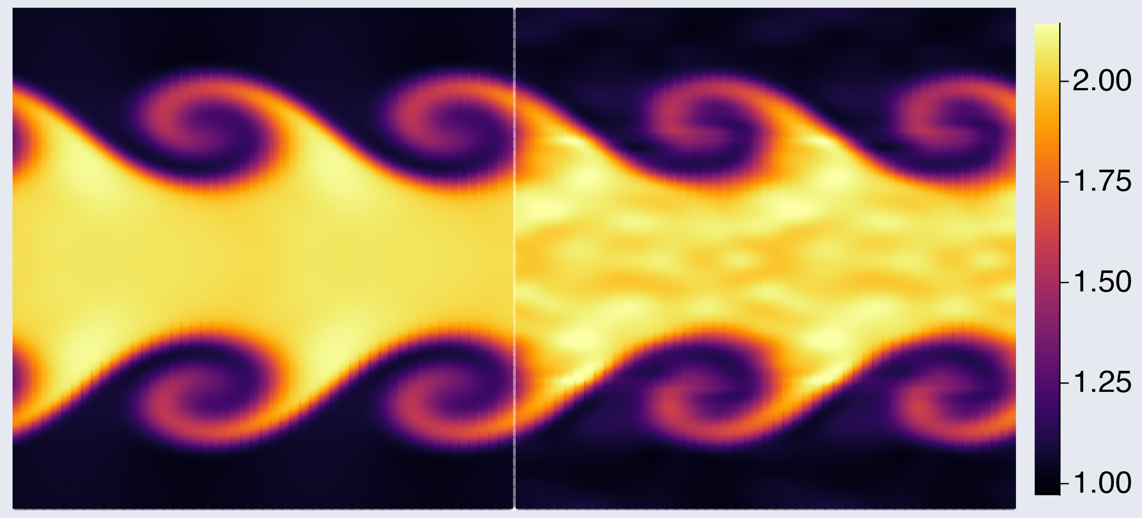


Figure: Density ρ at $T = 3.0$ in Kelvin-Helmholtz instability simulation: FOM (left) vs. ROM (125 modes, right).

- Smoothed Kelvin-Helmholtz instability: 2D compressible Euler equations on a periodic domain $[-1, 1]^2$ of 50×50 elements and polynomial degree $N = 4$.
- We add viscosity with $\epsilon = 1e-3$, run the simulation until $T = 3.0$, and use 100 snapshots enriched with entropy variables for POD modes. We employ 125 modes for ROM, which remains stable despite under-resolution.

Conclusion and Acknowledgement

- We present an entropy stable reduced order modeling of nonlinear conservation laws based on high order DG methods.
- We develop structure-preserving hyper-reduction techniques which preserve entropy stability.

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References

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